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General relativistic number effects in gyroscopes

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Abstract. A relativistic effect involving the local number density of particles on a rotating contour is predicted from the requirement that the total number of particles is constant. The effect is conjugate to that resulting from the invariance of the action or phase integral and is observable through the phenomenon of unipolar induction.

1. Introduction

In some previous publications (Forder 1984a, b), it was demonstrated how various types of gyroscope could be analysed by the method of adiabatic invariance. By requiring that the action integral $I = \oint p \, dq$ for each particle on the gyro contour should be a constant, the behaviour of the particles under the influence of inertial rotations and other general relativistic fields could be determined. No account was taken, however, of the invariance of the number of particles confined to the contour, a necessary requirement, even for massless particles, if the total action of the system is to be constant.

The purpose of this paper is to show that such an invariance leads to a further relativistic effect in gyroscopes involving the number density of particles. The effect is, therefore, conjugate to the more familiar gyroscopic effects which concern the momentum (i.e. action density) of the particles. When considering particles which are confined to a ring in the form of a macroscopic quantum state, the effect should be observable through the amplitude of the wavefunction rather than through its phase. Classically, however, the effect is responsible for the phenomenon of unipolar induction and is of relevance to the long-standing controversy regarding the origin of the electromotive force produced by rotating magnetic sources.

2. The number function

We begin by considering a spatially closed contour (figure 1) on which are confined a number of otherwise free identical particles whose motion is described by Hamilton's function $S(x^\mu)$. Values of S at neighbouring points in spacetime $x^\mu = (x^0, x^i)$† are therefore determined by the four-momentum p_μ of each particle:

$$p_\mu = -\partial S / \partial x^\mu. \qquad (1)$$

† Greek indices will be used to denote spacetime coordinates and run from 0–3. Latin indices denote spatial coordinates only and run from 1–3.

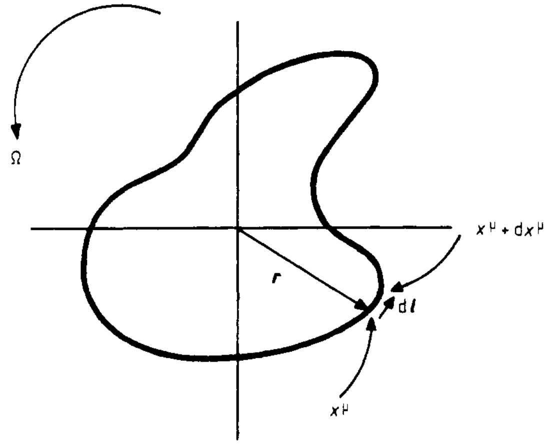

Figure 1. The model gyroscope. Particles circulate on the contour in an anticlockwise direction at a rate j and with line density n. In changing his spacetime coordinates by dx^{μ}, an observer on the contour passes (i.e. counts) a total number of particles $dN = (\partial N/\partial x^{\mu})\, dx^{\mu} = n\, dl - j\, d\tau$.

In the absence of general relativistic effects, we may consider such derivatives in terms of a single coordinate x (representing distance measured along the contour from some origin) and the usual time coordinate $t = x^0/c$ and write

$$dS = p\, dx - E\, dt \tag{2}$$

where $E = -\partial S/\partial t$ is the particle energy and $p = \partial S/\partial x$ is its momentum.

Whilst the function S describes the motion of a given particle, information concerning the distribution of many such particles in spacetime may be obtained from an analogous scalar function $N(x^{\mu})$, the number function. As with S, only the differences in N have physical significance, since the absolute number of particles counted by an observer clearly depends on when and where he begins. We define, therefore, a density of particles, D_{μ}, analogous to the action density of equation (1), where

$$D_{\mu} = -\partial N/\partial x^{\mu}. \tag{3}$$

Again, in the absence of relativistic effects, we may write the difference dN between neighbouring points in spacetime as

$$dN = n\, dx - j\, dt \tag{4}$$

where $j = -\partial N/\partial t$ is the rate at which particles pass a stationary observer and $n = \partial N/\partial x$ is their local density on the contour. In practice, dN then represents the number of particles counted by an observer who moves a distance dx in a time dt and who counts particles that pass him in a clockwise direction as positive and those that pass anticlockwise as negative.

If the particles on the contour constitute a macroscopic quantum state the amplitude and phase functions of the one-dimensional wavefunction $\psi = \psi_0 \exp(i\phi)$ are given in terms of n and S as

$$\psi_0 = n^{1/2} \tag{5}$$

$$\phi = S/\hbar = kx - \omega t \tag{6}$$

where $E = \hbar\omega$ and $p = \hbar k$. Points of constant phase must therefore propagate with the velocity

$$v_\phi = \omega/k = E/p \tag{7}$$

whilst from equation (4), points of constant number propagate with the particle velocity

$$v = j/n. \tag{8}$$

3. Measurements on a rotating contour

Since the quantities defined by equations (1) and (3) are derivatives with respect to coordinates, they are not, in general, the results of measurements made by an observer on the contour using clocks and measuring rods. As discussed elsewhere (Forder 1984a), in order to determine the results of such measurements it is necessary to consider the changes in S (or ϕ) and N in terms of the corresponding intervals of proper length $\mathrm{d}\boldsymbol{l}$ and proper time $\mathrm{d}\tau$. Such intervals will be determined by the nature of the local spacetime metric $g_{\mu\nu}$ and hence the presence of any gravitational fields. In this paper, however, we restrict ourselves to considering the effects of a uniform inertial rotation rate $\boldsymbol{\Omega}$ where the intervals are related to the coordinate differences according to

$$(\mathrm{d}\boldsymbol{l})_i \approx \mathrm{d}x^i \tag{9}$$

$$\mathrm{d}\tau = \mathrm{d}x^0/c + \mathrm{d}\theta \approx \mathrm{d}x^0/c - (\boldsymbol{\Omega} \wedge \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l}/c^2 \tag{10}$$

where $\boldsymbol{r}$ is the position vector of the observer and we assume $\Omega r \ll c$.

Following previous arguments, we consider the difference $\mathrm{d}S$ between neighbouring spacetime points in the form:

$$\mathrm{d}S = (\partial S/\partial x^\mu)\,\mathrm{d}x^\mu = p\,\mathrm{d}l - E\,\mathrm{d}\tau \qquad (\text{sum over } \mu = 0, 1, 2, 3) \tag{11}$$

and then, by separating spatial and temporal parts of the expression, we obtain, using equations (9) and (10),

$$(\partial S/\partial x^i)\,\mathrm{d}x^i \approx p\,\mathrm{d}l - E\,\mathrm{d}\theta \qquad (\text{sum over } i = 1, 2, 3) \tag{12}$$

where p and E are the proper (i.e. measured) values of momentum and energy. In the same way, we may write for the number function:

$$(\partial N/\partial x^i)\,\mathrm{d}x^i \approx n\,\mathrm{d}l - j\,\mathrm{d}\theta \qquad (\text{sum over } i = 1, 2, 3) \tag{13}$$

where n and j are the proper values of particle density and particle rate, respectively.

Integration of equations (12) and (13) around the contour then determines the action I_0 of the motion and the total number of particles involved, N_0. For simplicity we consider situations where the motion and distribution of the particles is uniform and write

$$I_0 = \oint (\partial S/\partial x^i)\,\mathrm{d}x^i = pL + E\Delta T \qquad (\text{sum over } i = 1, 2, 3) \tag{14}$$

$$N_0 = \oint (\partial N/\partial x^i)\,\mathrm{d}x^i = nL + j\Delta T \qquad (\text{sum over } i = 1, 2, 3) \tag{15}$$

where $L = \oint dl$ is the length of the contour and $\Delta T = -\oint d\theta$ is the Sagnac clock-synchronisation discrepancy, given in terms of the contour area S and rotation rate $\boldsymbol{\Omega}$ as (Landau and Lifshitz 1971)

$$\Delta T = 2\boldsymbol{\Omega} \cdot \boldsymbol{S}/c^2. \tag{16}$$

Equation (14) therefore relates the measured values of particle energy and momentum (E and p) to the total action I_0 whilst equation (15) is the analogous expression relating measured values of particle rate and density (j and n) to the total number of particles N_0. Only in an inertial frame where $\Delta T = 0$ can we write $I_0 = pL$ and $N_0 = nL$.

4. The invariance of action and number

The invariance of the action (or phase) integral I_0 (equation (14)) of particles confined to a rotating contour, and its consequences in determining the behaviour of gyroscopes, has been fully considered elsewhere (Forder 1984a, b). Although the invariance of the total number of particles N_0 on the contour is not difficult to understand, particularly when the particles involved possess rest mass, its consequences are no less important. By equating the values of I_0 and N_0 which apply when the contour is at rest ($\Delta T = 0$) to those when it is rotating, it is possible to derive expressions for the local changes in particle motion which an observer on the contour will detect. Thus, if, when initially at rest in an inertial frame, $I_0 = (p - \Delta p)L$ and $N_0 = (n - \Delta n)L$, the changes in momentum Δp and particle density Δn brought about by applying a given angular velocity to the contour are, using equations (14) and (15),

$$\Delta p = -E\Delta T/L = -2E\boldsymbol{\Omega} \cdot \boldsymbol{S}/Lc^2 \tag{17}$$

$$\Delta n = -j\Delta T/L = -2j\boldsymbol{\Omega} \cdot \boldsymbol{S}/Lc^2. \tag{18}$$

Equation (17) has, of course, been derived elsewhere whilst equation (18) predicts the analogous relativistic effect involving the particle density on the contour.

A circular contour of radius $R = 2S/L$ rotated about an axis through its centre and normal to its plane is of particular interest since the changes are then simply

$$\Delta p = -\Omega ER/c^2 \tag{19}$$

$$\Delta n = -\Omega jR/c^2. \tag{20}$$

It is perhaps not unnecessary to remark that, in this case, the pairs of local variables (p, E) and (n, j) as measured in the rotating frame can be Lorentz-transformed back to the inertial frame to obtain their original values:

$$p - \Delta p = \gamma(p + \Omega ER/c^2) \tag{21}$$

$$n - \Delta n = \gamma(n + \Omega jR/c^2) \tag{22}$$

where $\gamma = (1 - \Omega^2 R^2/c^2)^{-1/2}$. When $\Omega R/c$ is small, γ is close to unity and equations (19) and (20) are obtained directly.

5. Excess charge and potential of rotating circuits

Equation (18) may be applied to a variety of physical systems but in the remainder of this paper we shall confine ourselves to the case of a perfectly conducting loop of wire carrying a current $I = qj$ (with $q = e$, the electronic charge). Although such a contour is electrically neutral in the sense that there are always equal numbers of electrons and lattice charges, equation (18) predicts that, when the contour is accelerated to an angular velocity $\boldsymbol{\Omega}$, a co-rotating observer will see an excess charge per unit length:

$$q\Delta n = -I\Delta T/L = -2I\boldsymbol{\Omega}\cdot\boldsymbol{S}/Lc^2 \tag{23}$$

or, in the case of a circular contour of radius R, equation (20),

$$q\Delta n = -\Omega RI/c^2. \tag{24}$$

This excess charge is, of course, only a local difference Δn in the densities of electrons and lattice charges and is purely a result of the difficulties with clock synchronisation in the non-inertial frame. It is, however, very real to the observer, not least of all because Gauss' law ensures that it has an associated electric field. For a circular loop constructed from wire of radius $a \ll R$, this field is directed radially outward from the surface of the conductor and is of magnitude

$$E = q\Delta n/2\pi\varepsilon_0 a = -\Omega RI\mu_0/2\pi a = -\Omega RB \tag{25}$$

where $B = \mu_0 I/2\pi a$ is the magnetic field at the surface and we have written $\mu_0 = (\varepsilon_0 c^2)^{-1}$.

The presence of such an electric field will also change the potential of the contour as seen in the rotating frame and, in terms of the capacitance per unit length of the loop ε_l, this new potential may be written as

$$V = q\Delta n/\varepsilon_l = -I\Delta T/C_0 = -\Omega RI\mu_l \tag{26}$$

where $C_0 = L\varepsilon_l$ is the total capacitance and $\mu_l = (\varepsilon_l c^2)^{-1}$ is the inductance per unit length. Note that since $\mu_l I$ is the magnitude of the effective vector potential $\boldsymbol{A}$ at each point on the contour in the rotating frame, the variables $(\mu_l I, V)$ can be Lorentz transformed back to the inertial frame (cf equations (21) and (22)) to give the original (zero) potential:

$$0 = \gamma(V + \Omega R\mu_l I). \tag{27}$$

In certain respects the excess charge produced by the inertial rotation is similar to the apparent volume charge density present within a medium of non-uniform electric polarisation. Indeed, as Heer (1964) has shown, these inertial effects can be analysed directly in terms of an anisotropic constitutive relation for free space:

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} - \varepsilon_0(\boldsymbol{\Omega}\wedge\boldsymbol{r})\wedge\boldsymbol{B} \tag{28}$$

corresponding to an apparent volume charge density (cf Schiff 1939, Trocheris 1949):

$$\begin{aligned}\Delta\rho &= \boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E} - \boldsymbol{D})\\ &= 2\varepsilon_0\boldsymbol{\Omega}\cdot\boldsymbol{B} - \varepsilon_0(\boldsymbol{\Omega}\wedge\boldsymbol{r})\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{B})\\ &= 2\varepsilon_0\boldsymbol{\Omega}\cdot\boldsymbol{B} - \boldsymbol{\Omega}\cdot\boldsymbol{r}\wedge\boldsymbol{J}/c^2\end{aligned} \tag{29}$$

where $\boldsymbol{J} = \boldsymbol{\nabla}\wedge\boldsymbol{B}/\mu_0$ is the current density. Integration of this expression over the highly localised volume of the current-carrying contour then yields equation (23). Note, however, that, according to this analysis, observers in the rotating frame who are not actually on the contour are also aware of a rotationally induced charge density, but one which is proportional to $\boldsymbol{B}$ rather than $\boldsymbol{J}$. When there are no 'real' charges, as with the rotating contour, the electric displacement $\boldsymbol{D}$ must vanish, producing a rotational-dependent electric field (cf equation (25)):

$$\boldsymbol{E} = (\boldsymbol{\Omega} \wedge \boldsymbol{r}) \wedge \boldsymbol{B}. \tag{30}$$

6. Unipolar induction and magnetic inertia

Equations (25) and (30) are of some interest since, although the magnetic field $\boldsymbol{B}$ which appears in the expressions is measured in the rotating frame, the force $q\boldsymbol{E}$ experienced by a rotating observer equipped with a test charge q is the same as if he were moving *relative* to an identical magnetic field with velocity $\boldsymbol{v} = \boldsymbol{\Omega} \wedge \boldsymbol{r}$. If, therefore, we wish to consider this effect using the convenient fiction of magnetic lines of force, it is necessary to endow such lines with an intrinsic inertia (derived from their finite self energy) which, by preventing them from acquiring the rotational motion of their source (the contour), keeps them fixed in the inertial frame.

Whilst this view of the system may be criticised as being picturesque rather than physically meaningful, the question as to whether magnetic fields do or do not 'rotate' with their source has, in fact, been the subject of heated debate, both in the past and also more recently (e.g. Djuric 1975) since its answer is of some relevance to identifying the source of electromotive force produced by unipolar induction (e.g. Rosser 1968). This phenomenon is most readily considered in terms of Faraday's homopolar generator, consisting of a suitable magnetic source, such as the conducting permanently magnetised disc of figure 2, rotating between stationary sliding contacts on its axis 0 and perimeter A. A meter connected between these contacts, together with the return

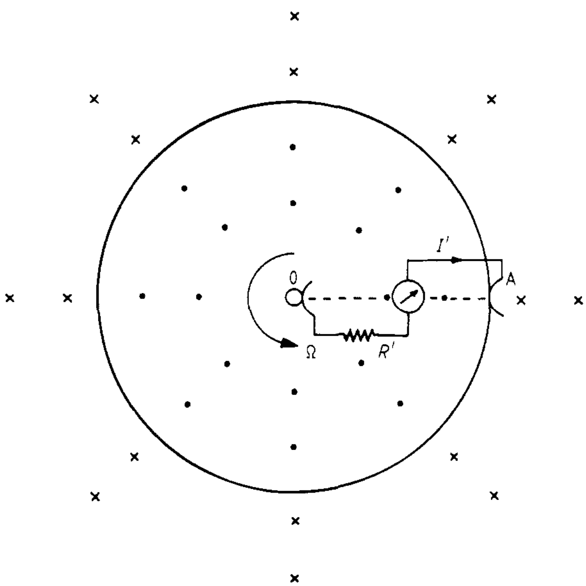

Figure 2. Faraday's homopolar generator. A magnetic source such as a permanently magnetised disc or a circular current loop is rotated between stationary contacts at 0 and A. The EMF generated is $\mathscr{E} = -\Omega\Phi_0/2\pi = R'I'$ where Φ_0 is the flux enclosed by the perimeter. Arrow heads (•) and tails (×) denote flux lines.

path through the disc (shown as a broken line), form a complete circuit, through which a current I' flows, of magnitude proportional to the rotation rate. The question then is: is the electromotive force that drives this current produced by rotating lines of force sweeping through fixed parts of the circuit or in the magnetised disc itself as it rotates in its own, stationary, field? Either hypothesis can be used to predict the correct magnitude and sense of the EMF as

$$\mathscr{E} = -\Omega\Phi_0/2\pi \tag{31}$$

where Φ_0 is the total magnetic flux linking the area of the disc. Curiously, however, there is scant experimental evidence to distinguish between the opposing views, apart from the few results of Pegram (1917). Although surrounded by some controversy, these results appear to support the view that the field does not rotate.

Whilst we do not attempt to provide an explanation of unipolar induction in the general case, we consider a modified form of homopolar generator, in which the permanent magnet of figure 2 is replaced by a simple metallic disc whose perfectly conducting perimeter carries a circulating current I. In the rotating frame, the perimeter of the disc appears to possess an excess charge (equation (24)) which, under the action of the potential difference of equation (26), will tend to flow towards 0. In the inertial frame, however, the perimeter is initially neutral and any charge that flows within the rotating frame must leave the disc with a deficiency as seen by a fixed observer. Hence, the charge that reaches 0 immediately returns to the perimeter through the external circuit and the continuous current I' is set up. Such behaviour, of course, originates from the fundamental inability of stationary and rotating observers to agree on the neutrality, or otherwise, of the perimeter of the disc.

In order to neglect the magnetic interaction between I' and the perimeter current I, we must assume a sufficiently large circuit resistance R' to ensure that I'/I is small. With this restriction, however, it follows that the source of the EMF is simply the rotationally induced potential of the perimeter in the rotating frame, equation (26). For a contour of length $L = 2\pi R$, this EMF is of the same magnitude as given by equation (31), namely

$$\mathscr{E} = -I\Delta T/C_0 = -\Omega L_0 I/2\pi = -\Omega\Phi_0/2\pi \tag{32}$$

where $\Phi_0 = L_0 I$ is the enclosed magnetic flux and $L_0 = \mu_l L$ is the total inductance of the contour. The resistance R' should be large enough to satisfy the condition

$$-I'/I = \Omega L_0/2\pi R' \ll 1. \tag{33}$$

7. Conclusions

By considering the invariance of the total number of particles N_0 confined to a rotating contour, the change in local number density Δn in a gyroscope of length L and area S can be obtained in the form:

$$\Delta N_0 = 0 = L\Delta n + j\Delta T \tag{34}$$

where j is the particle rate and $\Delta T = 2\mathbf{\Omega}\cdot\mathbf{S}/c^2$ is the Sagnàc synchronisation discrepancy.

When considering the charged particles in a conducting ring carrying a circulating current I, the change in particle density results in an increased potential in the rotating frame:

$$V = -I\Delta T/C_0 = -\Omega L_0 I/2\pi \tag{35}$$

where C_0 is the capacitance of the ring and L_0 is its inductance. The result can be applied to an idealised form of Faraday's homopolar generator and shows that the EMF within such a system originates in the rotating frame.

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